for the $2s$ excitation, and

$$\Psi_{LM}{}^{1d}=\sum_{L'}(L[\![L'2)\times\{\psi_{L'M'}[(1s)^3(1p)^{A-5}]_8\,\Phi_{2m}(1_8 2)\}_{LM} \quad (8)$$

for the $1d$ excitation. The $(L[\![L'2)$ are coefficients of fractional parentage, and the curly brackets denote antisymmetric vector coupled wave functions. The spin and isobaric spin quantum numbers are not written explicitly, but their incorporation into Eqs. (7) and (8) is always implied through the orbital angular momentum (in other words L symbolically stands for $LSJT$).

It is easy to demonstrate that spurious center-of-mass excitations are absent from the shell model wave functions in Eqs. (7) and (8). Following closely the analysis of Elliott and Skyrme,[5] we may write either (7) or (8) in the form:

$$\Psi_{LM}=P_{A-3}(\mathbf{q}_i)\exp[-\tfrac{1}{2}\sum q_i{}^2], \quad (9)$$

where $P_{A-3}(\mathbf{q}_i)$ is a polynomial of degree $A-3$ in the coordinates $\mathbf{q}_i$ and is antisymmetric under the simultaneous interchange of orbital, spin, and isobaric spin coordinates of any two nucleons. The identities ($\mathbf{Q}=\sum\mathbf{q}_i/A$):

$$\sum_i q_i{}^2=\sum_i(\mathbf{q}_i-\mathbf{Q})^2+AQ^2 \quad (10)$$

and

$$P_{A-3}(\mathbf{q}_i-\mathbf{Q})=P_{A-3}(\mathbf{q}_i) \quad (11)$$

must be established to complete the proof. Equation (10) is obvious. Equation (11) may be proved by expanding $P_{A-3}(\mathbf{q}_i-\mathbf{Q})$ into a power series in Q:

$$P_{A-3}(\mathbf{q}_i-\mathbf{q})=P_{A-3}(q_i)+\mathbf{Q}\cdot P_{A-4}(q_i)+\cdots. \quad (12)$$

In order that Eq. (12) retain its homogeneity, one of the oscillator orbitals in $P_{A-4}(\mathbf{q}_i)$ must be demoted to a lower energy state. The $\varphi_{200}(\mathbf{q}_-)$ and $\varphi_{12m}(\mathbf{q}_-)$ orbitals are now invariant under $\mathbf{q}_i\rightarrow\mathbf{q}_i-\mathbf{Q}$ so we must demote a $1p$ orbital. The $1s$ shell is filled, however, with three $1s$ nucleons and $\varphi_{100}(\mathbf{q}_+)$; thus we must have $P_{A-4}(\mathbf{q}_i)\equiv 0$ and in a like manner

$$P_{A-n}(\mathbf{q}_i)\equiv 0 \quad \text{if} \quad n>3. \quad (13)$$

The wave functions in Eqs. (7) and (8) are therefore free of spurious center-of-mass excitation, and appropriate for shell-model calculations.

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Nuclear Spectroscopy With a Soft Core Potential*

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Nuclear spectroscopic calculations with a potential of the type $v(r)=v_0[(r^n-c^n/r^n)]\exp(-r^2/r_0{}^2)$ are made for the core radius $c=0.4$ F and potential range $r_0=2.06$ F. For simplicity of calculation, $n=2$ is assumed. The calculations are made for f, p, and g-shell nuclei. It is shown that in the case of Zr^{90} the configuration dependence of the effective two-body interaction can be replaced by a singular potential. The configuration interaction in this nucleus is found to depress the ground state 0^+ by ~ 0.4 MeV and raise the excited 0^+ level by the same amount. For the f and p shells, the level splittings in Ti^{50}, Ni^{58}, and Be^{10} are analyzed. The calculations show that the triplet forces are small for these configurations.

I. INTRODUCTION

IT has been shown by many authors that the nucleon-nucleon interaction inside a nucleus is strongly attractive and is believed to have a hard core at small distances of order 0.4–0.5 F. One, therefore, cannot use a perturbation approach. Also, the actual wave functions of the individual nucleons are modified due to the strong internucleonic force. In calculations one should therefore use such modified wave functions in order to calculate various quantities. However, if one uses the perturbation theory one could also use another approach. Instead of using the modified form of the wave functions, one could use the unperturbed shell-model wave functions and the modified form of the two-body Hamiltonian. Very little is known about the exact nature of the nucleon-nucleon interaction. However, some elegant theories[1–3] which would provide better understanding of the nature of nuclear forces are now available. In the past few years, the Brueckner[1] many-body theory has been extensively used in deducing various quantities concerned with the Weizsäcker mass formula. This theory is based upon the assumption of the two-particle correlations. A "reaction matrix" for a two-body potential is evaluated

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[3] H. Feshbach and E. Lomon, Ann. Phys. (N. Y.) (to be published).

in a self-consistent manner and an average binding energy is calculated in the case of nuclear matter. The calculations are then further extended[4–6] with the local density approximation to the case of a finite nucleus. Furthermore, it is difficult to evaluate the reaction matrix exactly when the potential has an infinite repulsive core. The problem of hard core has been examined by several authors. It is shown by Moszkowski[7] that a nonsingular velocity-dependent potential can well be introduced to replace the singularity of the hard core. It is also shown[7] that the elastic scattering data alone even up to 300 MeV do not appear to be sufficient to specify the nucleon-nucleon potential accurately enough for purposes of many-body calculations and that the use of binding energies of complex nuclei may prove helpful to specify this interaction precisely.

In order to avoid the complexities due to the hard core, some approximation methods[8,9] have also been suggested. It is nonetheless obvious that even an effective potential acting among the nucleons in a shell-model nucleus might possess a strong repulsive character. This has recently been shown by Pandya[10] on the basis of purely simple shell-model calculations. Some calculations, with the so called "realistic potentials," have also been recently made. For these calculations, we refer to the paper of Dawson and Walecka[11] who have used Gammel-Thaler potential in describing some of the features of light nuclei. Their prediction about the $p_{1/2}-p_{3/2}$ single-particle level difference as $\sim$5–6 MeV gives results[12] which are in good agreement with the experiments. However, an interesting conclusion by Blatt[13] regarding a 'new potential' (e.g., Hamada-Johnston or Breit potential) and the scattering data is that as the agreement between the theory and the experiment improves, the deviations from the He^3 binding energy become more serious. In view of this, we make calculations in this paper for $T=1$ states of various nuclei, still in the framework of the shell model, with a nucleon-nucleon potential having a soft core. In Sec. II, the method of calculating various matrix elements is presented. In Secs. III and IV, the method developed in Sec. II is applied for the f, p, and g-shell nuclei. In Sec. V, we discuss some of the aspects of the potential we choose and in Sec. VI we make some overall comments regarding our approach.

II. METHOD OF CALCULATIONS

The method of evaluating the matrix elements of vatious shell-model states is essentially the same as that used earlier.[12] We only write the final expression for the matrix elements of a two-body interaction as[14]

$$\langle j_1 j_2 JM|H_{12}|j_1' j_2' JM\rangle$$

$$=aa'\sum_{LsN\lambda nl} A\begin{Bmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{Bmatrix} A\begin{Bmatrix} l_1' & s_1' & j_1' \\ l_2' & s_2' & j_2' \\ L & S & J \end{Bmatrix}$$

$$\times B_{nl n\lambda}{}^{n_1 l_1 n_2 l_2} B_{nl N\lambda}{}^{n_1' l_1' n_2' l_2'}[1+(-)^{l+S}]^2 I_{nl}, \quad (1)$$

where a and a' equal $\frac{1}{2}$ if the particles are equivalent, and equal $1/\sqrt{2}$ if the particles are inequivalent. The integrals I_{nl} are

$$I_{nl}=\langle nl,S||H_{12}||nl,S\rangle=\sum_l f_l I_l, \quad (2)$$

where f_l are the coefficients corresponding to the relative orbital angular momentum l and

$$I_l=\int_0^\infty R_l^2(r)v(r)dr, \quad (3)$$

where $\Psi_l(r)=R_l(r)/r$ are the harmonic oscillator wave functions. The general form for the central two-body interaction can be written as

$$H_{12}=(A_W W+A_M M+A_B B+A_H H)V(r), \quad (4)$$

where A_W, A_M, A_B, and A_H are the coefficients corresponding to the various mixtures of the Wigner (W), Majorana (M), Bartlett (B), and Heisenberg (H) forces. Substituting the well-known expressions for W, M, B, and H in terms of the spin and isotopic spin states, one obtains

$$H_{12}{}^T=(a+b\sigma_1\cdot\sigma_2)v^T(r), \quad (5)$$

where a and b depend upon the coefficients in (4) as well as the isotopic spin T which is 0 for $J=$ odd and 1 for $J=$ even. These parameters a and b are to be evaluated from the experimental results. For $T=1$ states, one can then define these coefficients in terms of the coefficients A_{TS} defined by Barker[15] as

$$A_{11}=W-M+H+B=(v_0)^{-1}(a+b), \quad (6)$$

$$A_{10}=W+M+H-B=(v_0)^{-1}(a-3b), \quad (7)$$

where v_0 is the strength of the potential $v(r_{12})$. In nuclear spectroscopy calculations one generally assumes a Yukawa, Gaussian, or an exponential form. Such

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[5] K. S. Masterson, Jr., and A. M. Lockett, Phys. Rev. **129**, 776 (1963).
[6] K. A. Brueckner, A. M. Lockett, and M. Rotenberg, Phys. Rev. **121**, 255 (1961).
[7] S. A. Moszkowski, Phys. Rev. **129**, 1901 (1963).
[8] S. A. Moszkowski and B. L. Scott, Ann. Phys. (N. Y.) **11**, 65 (1960); H. S. Köhler, Ann. Phys. (N. Y.) **16**, 375 (1961).
[9] H. A. Bethe, B. H. Brandow, and A. G. Petschek, Phys. Rev. **129**, 225 (1963).
[10] S. P. Pandya, Nucl. Phys. **43**, 636 (1963).
[11] J. F. Dawson and J. D. Walecka, Ann. Phys. (N. Y.) **22**, 133 (1963).
[12] Y. R. Waghmare, Phys. Rev. **134**, B1185 (1964).
[13] J. M. Blatt and L. M. Delves, Phys. Rev. Letters **12**, 566 (1964).
[14] S. K. Shah and S. P. Pandya, Nucl. Phys. **38**, 420 (1962).
[15] F. C. Barker, Phys. Rev. **122**, 572 (1961).

calculations have been made by various authors. It is recently shown by Warke and Waghmare[16] that for the same nucleus, with a given effective range and strength, all these three potentials give essentially the same results. However, we are interested in a potential which will have a repulsive core at small distances. The potential that one generally uses in fitting the scattering data is of the form

$$v(r)=+\infty\,, \quad r\leq c$$
$$=v_{\text{attr}}(r)\,, \quad r>c\,, \tag{8}$$

where c is the core radius of order 0.4–0.5 F, and $v_{\text{attr}}(r)$ is the attractive part of the nucleon-nucleon potential and is usually of the Yukawa shape $v(r)=e^{-\mu r}/\mu r$, with $\mu=1.4$ F. This potential along with the square-well potential is illustrated in Fig. 1. It is, however, known that the calculations with potentials of the type described in (8) are difficult. In order to simplify the calculations, we prefer to choose an analytic form which would show the soft core behavior. Such a form can be taken to be

$$v(r)=\left(\frac{r^2-c^2}{r^2}\right)e^{-(r/r_0)^2}. \tag{9}$$

This potential is also illustrated in Fig. 1. Substituting expression (9) in (3) we obtain for the integrals I_l

$$I_l=\eta_l^{l+3/2}\{1-[2\zeta/\eta_l(2l+1)]\} \tag{10}$$

with

$$\eta_l=\lambda^2/(1+\lambda^2)\,, \quad \lambda=r_0/\sqrt{2}r_l\,, \quad \zeta=c^2/r_l^2\,,$$

where r_0 is the range of the potential and r_l is the range

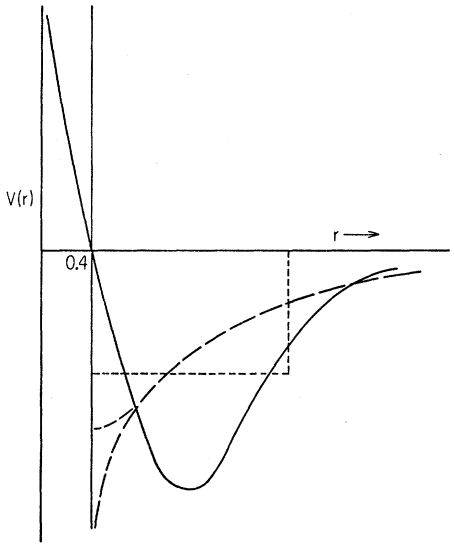

FIG. 1. Illustrative diagram of various potentials which are generally used. The dashed curve shows the characteristic shape of Yukawa, Gaussian, or exponential potential; the dotted curve shows the square well; and the solid curve represents the potential described by Eq. (6).

[16] C. S. Warke and Y. R. Waghmare, Phys. Rev. **135**, B872 (1964).

TABLE I. Table of integrals I_l ($c=0.4$, $r_0=2.01$) defined by Eq. (10) for $\eta_l=1$ to 4.

I_l \ η_l	1	2	3	4
I_0	0.053665	0.120498	0.202387	0.296986
I_1	0.015503	0.044913	0.094448	0.167349
I_2	0.003292	0.013999	0.038859	0.085560
I_3	0.000675	0.004267	0.015728	0.043184
I_4	0.000137	0.001292	0.006332	0.021704
I_5	0.000028	0.000389	0.002543	0.010887
I_6	0.000006	0.000118	0.001020	0.005456
I_7	0.000001	0.000035	0.000409	0.002733
I_8	0.000000	0.000011	0.000164	0.001368

of the harmonic oscillator wave function as described previously. It is clear that as $\zeta\to 0$, I_l go over to the integrals for a pure Gaussian shape. The integrals I_{nl} in (2) are tabulated for a Gaussian potential for $n=0$ to 4, $l=0$ to 5 earlier.[12] In Table I, we tabulate the integrals I_l for various values of l and η_l.

III. ANALYSIS OF THE f AND p SHELLS

We now apply the method developed in the previous section to the f, p, and g shells. We consider the energy levels of a nucleus with ground state configuration as $(f_{7/2})^2$, e.g., Ti^{50}. In order to restrict the number of parameters occurring in the theory, we choose the core radius to be 0.4 F as deduced by Brueckner and others. We also keep the range of the two-body potential r_0 to be fixed at 2.07 F from our earlier analysis.[12] [In this paper there is an error in defining r_0. r_0 given in Table II of this paper should be $r_0/\sqrt{2}$. The value for $r_0=1.47$ F then comes close to the one calculated in Ref. 17.] We are then left with only two parameters, namely a and b, to be determined from the experimental information. The low-lying levels of Ti^{50} are shown in Fig. 2. These levels can arise from $(f_{7/2})^2$ configuration. In order to evaluate the parameters a and b appearing in the Hamiltonian (5), we assume that the levels 0^+, 2^+, 4^+, and 6^+ of Ti^{50} arise from a pure configuration $(f_{7/2})^2$. As the parameters a and b appear in calculations in combinations of $(a+b)$ and $(a-3b)$, we define two other parameters

$$u=(a-3b)$$

and

$$v=(a+b)$$

corresponding to the spin states $s=0$ and $s=1$, respectively. We choose three levels 0^+, 2^+, and 6^+ in order to evaluate the two parameters u and v. From these levels the parameters u and v are evaluated for various values of λ. The value of λ for Ti^{50} is ~ 0.8. For this value of λ, $u\sim -55$ MeV and $v\sim -7$ MeV (Table II). In order to compare our results with those obtained by other authors, we fix the value of A_{10} at 0.6. The quantities v_0 and A_{11} then are -91.7 MeV and 0.08,

[17] V. K. Thankappan, Y. R. Wagmare, and S. P. Pandya, Progr. Theoret. Phys. (Kyoto) **26**, 22 (1961).

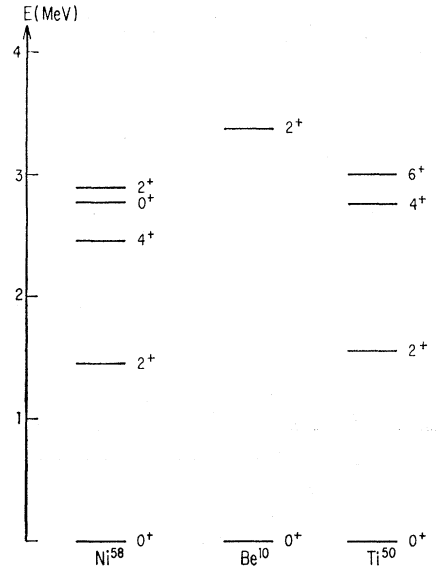

FIG. 2. The low-lying experimental levels of Ti^{50}, Be^{10}, and Ni^{58}.

respectively. With these values of the parameters, the 4^+ level in Ti^{50} is placed at 2.40 MeV. The observed value is 2.76 MeV.

For Ni^{58} the value of λ is 0.77 and its ground-state configuration is $(p_{3/2})^2$. We assume that the nature of the effective two-body interaction does not change much from Ti^{50} to Ni^{58}. In that case we can take the values of the parameters u and v for $\lambda=0.77$ from the analysis of Ti^{50}. The values of these parameters in terms of v_0 and A_{11} are given in Table II. The 2^+-0^+ splitting of the $(p_{3/2})^2$ configuration in Ni^{58} for this set of parameters is 1.80 MeV. The observed value is 1.45 MeV. It should be mentioned that in our calculations we did not include the configuration mixing from other excited levels in Ni^{58}. However, preliminary analysis shows that both 0^+ and 2^+ are depressed by almost the same amount so that the 2^+-0^+ splitting remains almost the same.

The nucleus Be^{10} also has the ground state configuration $(p_{3/2})^2$. Its energy is $1\hbar\omega$ less than the one in Ni^{58}. In view of earlier indications[12] that the effective interaction does not change its character in the same shell, we once again take the values of u and v for ~ 0.98 for Be^{10} from the analysis of Ti^{50} and Ni^{58}. The values of v_0 and A_{11} for $\lambda\sim 0.98$ are given in Table II. Then the 2^+-0^+ splitting in Be^{10} is obtained to be 3.52 MeV. The observed value is 3.37 MeV. It should be remarked that the $p_{1/2}-p_{3/2}$ separation by Dawson and Walecka[11] is ~ 6.0 MeV. This means that the ground state 0^+ as well as the excited state 2^+ in Be^{10} are almost pure. It is interesting to note from Table II that the parameters A_{11} vary from 0.11 to -0.02 indicating that the triplet forces operate weakly in these nuclei. A similar situation is observed even in the case of heavier nuclei in the $g_{9/2}$-shell as we shall see in the following section.

TABLE II. Table showing the comparison of the calculated and the observed results for various nuclei.

Nucleus	Level separation	λ	Calculated value in MeV	Observed value in MeV	$-v_0$ in MeV	A_{11}
Ti^{50}	4^+-0^+	0.80	2.40	2.76	92	0.08
Be^{10}	2^+-0^+	0.98	3.52	3.37	300	-0.02
Ni^{58}	2^+-0^+	0.77	1.80	1.45	175	0.11

IV. ANALYSIS OF THE g SHELL

The observed levels of Zr^{90} are shown in Fig. 3. The calculations on the energy levels of Zr^{90} have been made by some authors.[18] The ground-state configuration of Zr^{90} is $(p_{1/2})^2$ and the excited states arise when one or both the particles in $p_{1/2}$ shell go to $g_{9/2}$ shell. In this nucleus, we evaluate the parameters u and v for $\lambda=0.7$ from the observed positions of 2^+, 4^+, and 8^+ levels. The values so obtained are

$$u=-90 \text{ MeV},$$
$$v=-0.5 \text{ MeV}.$$

These values give for the triplet mixture $A_{11}\sim 0.003$ (Table II). With these values of the parameters, the separation of the 6^+ level and the 8^+ level is calculated to be 0.18 MeV. The observed value is 0.14 MeV. It is of interest to see that the triplet forces are very small.

In order to estimate the effect of configuration mixing, we evaluate the position of the ground state 0^+. This will be depressed to some extent from its unper-

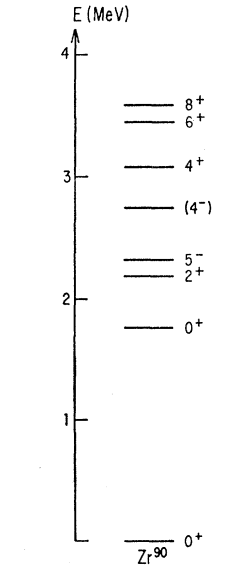

FIG. 3. Experimental energy levels of Zr^{90}.

[18] See, e.g., the list in Ref. 17.

turbed position due to the interaction of the first excited 0^+ level [arising from the $(g_{9/2})^2$ configuration]. We assume the $p_{1/2}-g_{9/2}$ separation to be 1.0 MeV. With the above parameters, the off-diagonal matrix element is estimated to be $\sim$0.85 MeV. The separation of the two 0^+ levels is calculated as 1.85 MeV while the observed value is 1.75 MeV (Table III). In fact, the configuration interaction depresses the ground state by $\sim$0.4 MeV from its unperturbed position while it raises the excited 0^+ level by the same amount. It is obvious that the configuration mixing is important in this nucleus.

We next calculate the positions of the 4^- and 5^- levels in Zr^{90}. 5^- is observed to be at 2.31 MeV while 4^- level is as yet not well established. The position of the 4^- level is once again $\sim$3.50 MeV. This value seems to be rather large as compared to the suspected position which is at $\sim$2.80 MeV. It is thus of interest to establish experimentally the position of the 4^- level. Another point of interest is the following.

It was shown by Thankappan *et al.*[17] that in order to explain the level structure of Zr^{90} nucleus on the basis of the simple shell-model calculations two sets of interaction were needed. In other words, an interaction which explains the levels of $(g_{9/2})^2$ configuration satisfactorily would not explain the ground state 0^+ or the 5^- levels. Their conclusion that the effective two-body interaction is configuration dependent is obvious. However, from the present analysis, it seems that such a configuration dependence can be simulated by means of a soft core potential. It may also be worthwhile to mention that the effect of using a velocity-dependent potential $v(p,r)=[p^2v(r)+v(r)p^2]$ would also be similar. We now make some remarks on the implications of the potential we have chosen for our analysis.

V. THE POTENTIAL

A potential which has a repulsive character at short distances has been suggested earlier by Goldhammer[19] for calculating the binding energies, quadrupole moments, etc. of O^{16}, H^2, H^3, and He^4 nuclei. However, it is recently shown by Ullah and Nesbet[20] that such a potential, with the core part separated from the attractive part, does not give satisfactory results for the binding energy of O^{16}. Another disadvantage of such a potential is that it has many more parameters and consequently is not convenient in the nuclear spectroscopic calculations. On the other hand, a potential with a simple analytic form with the required physical behavior would be more suitable. A family of such potentials could be mentioned.

$$V(r)=v_0[(r^n-c^n)/r^n]V(r_{12}), \qquad (12)$$

[19] P. Goldhammer, Phys. Rev. **116**, 676 (1959).
[20] N. Ullah and R. K. Nesbet, Phys. Rev. **134**, B308 (1964).

TABLE III. Observed and calculated energy levels of Zr^{90}.

J	0^+	0^+	2^+	5^-	(4^-)	4^+	6^+	8^+
E_{exptl} (MeV)	G.S.	1.75	2.18	2.31	2.80	3.08	3.45	3.59
E_{calc} (MeV)	G.S.	1.85	2.28	2.30	3.50	3.18	3.51	3.69

where $V(r_{12})$ may have any of the following shapes,

$$\begin{aligned} &\text{Gaussian:} && V(r_{12})=e^{-(r/r_0)^2} \\ &\text{Yukawa:} && V(r_{12})=e^{-r/r_0}/r/r_0 \\ &\text{Exponential:} && V(r_{12})=e^{-r/r_0}. \end{aligned} \qquad (13)$$

A suitable potential could then be chosen by a proper choice of n and r_0. There is also an advantage from the viewpoint of many-body calculations. By studying the potential in (12) for various values of the core radius a good understanding of the saturation density could be obtained. Due to the soft core, the many-body calculations may not be difficult for the application of the perturbation treatment.

VI. DISCUSSION AND CONCLUSION

Most of the calculations made in nuclear spectroscopic studies are based upon the pure attractive character of the two-nucleon potential. It is however now known that this may not be so and that such a potential may have a repulsive part at short distances. From the calculations made in Secs. III and IV it is evident that a repulsive interaction with soft core does provide an understanding of the effective nucleon-nucleon interaction. The method can also be applied to estimate the amount of admixtures in the wavefunctions due to the interactions of levels of same spin and parity but arising from different configurations. This we did in the case of Zr^{90} and observed that our potential partly replaces the configuration dependence observed earlier in Ref. 17. For Ni^{58}, where the calculated value is slightly higher than the observed one, the configuration mixing effect of the second 2^+ excited state may depress the first 2^+ state to a reasonable agreement. However, it is clear from Table II that triplet forces are comparatively small and the neglect of tensor forces would not introduce serious error. This conclusion about the tensor force was also independently arrived at by Goldhammer.[18] It is obvious that due to its intrinsic character, the potential described by Eqs. (12) and (13) would be valuable for further calculations in the study of many-body properties.

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